

SHARP EMBEDDING RELATIONS BETWEEN LOCAL HARDY AND α -MODULATION SPACES

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ABSTRACT. We give the optimal embedding relations between local Hardy space and α -modulation spaces, which extend the results for the embedding relations between local Hardy and modulation spaces obtained by Kobayashi, Miyachi and Tomita in [Studia Math. 192 (2009), 79-96].

1. INTRODUCTION

The modulation space $M_{p,q}^s$ was first introduced by Feichtinger [1] in 1983. Some motivations and historical remarks can be founded in [2]. As function spaces associated with the uniform decomposition (see [17]), modulation space has a close relationship with the topics of time-frequency analysis (see [6]), and it has been regarded as a appropriate space for the study of partial differential equations (see [19]). On the other hand, it is well known that the Besov space $B_{p,q}^s$, constructed by the dyadic decomposition on frequency plane, is also a popular function space in the studies of harmonic analysis and partial differential equations.

A more general function space, named α -modulation space, denoted by $M_{p,q}^{s,\alpha}$, was introduced by Gröbner [5] in 1992, in order to link modulation and Besov spaces by the parameter $\alpha \in [0, 1]$. The modulation spaces $M_{p,q}^s$ are the special α -modulation spaces when $\alpha = 0$, and the (inhomogeneous) Besov space $B_{p,q}^s$ can be regarded as the limit case of $M_{p,q}^{s,\alpha}$ as $\alpha \rightarrow 1$ (see [5]). Sometimes, we use $M_{p,q}^{s,1}$ to denote the inhomogeneous Besov space $B_{p,q}^s$ for convenience. The interested reader can find some basic properties of α -modulation spaces in [10]. We note that the α -modulation space is not the intermediate space between modulation and Besov spaces in the sense of complex interpolation (see [8]).

Embedding relations among frequency decomposition spaces, including Lebesgue spaces, Sobolev spaces, Besov spaces, modulation spaces and α -modulation spaces, have been concerned by many authors, for example, one can see Gröbner [5], Okoudjou [14], Toft [16], Sugimoto-Tomita [15], Wang-Han [10], Guo-Fan-Wu-Zhao [9] for embedding relation among modulation spaces, Besov spaces and α -modulation spaces. In particular, Kobayashi-Miyachi-Tomita [12] studied the embedding relations between local Hardy spaces h_p and modulation spaces $M_{p,q}^s$, Kobayashi-Sugimoto [13] consider the embedding relations between Sobolev and modulation spaces. We state their results as follows.

For $(p, q) \in (0, \infty]^2$, we define $A(p, q) = 0 \wedge n(1 - 1/p - 1/q) \wedge n(1/p - 1/q)$, $B(p, q) = 0 \vee n(1 - 1/p - 1/q) \vee n(1/p - 1/q)$, that is,

$$A(p, q) = \begin{cases} 0, & \text{if } (1/p, 1/q) \in A_1 : 1/q \leq (1 - 1/p) \wedge 1/p, \\ n(1 - 1/p - 1/q), & \text{if } (1/p, 1/q) \in A_2 : 1/p \geq (1 - 1/q) \vee 1/2, \\ n(1/p - 1/q), & \text{if } (1/p, 1/q) \in A_3 : 1/p \leq 1/q \wedge 1/2; \end{cases}$$

$$B(p, q) = \begin{cases} 0, & \text{if } (1/p, 1/q) \in B_1 : 1/q \geq (1 - 1/p) \vee 1/p, \\ n(1 - 1/p - 1/q), & \text{if } (1/p, 1/q) \in B_2 : 1/p \leq (1 - 1/q) \wedge 1/2, \\ n(1/2 - 1/q), & \text{if } (1/p, 1/q) \in B_3 : 1/p \geq 1/q \vee 1/2. \end{cases}$$

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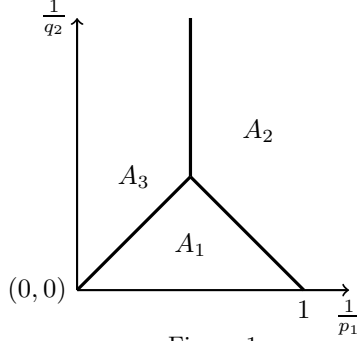


Figure 1

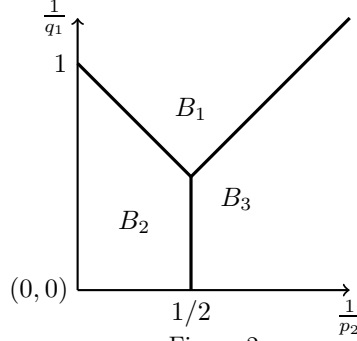


Figure 2

Theorem A (cf. [12]) Let $0 < p \leq 1$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then, $h_p \subset M_{p,q}^s$ if and only if $s \leq A(p, q)$ with strict inequality when $1/q > 1/p$.

Theorem B (cf. [12]) Let $0 < p \leq 1$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then, $M_{p,q}^s \subset h_p$ if and only if $s \geq B(p, q)$ with strict inequality when $1/p > 1/q$.

Theorem C (cf. [13]) Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. Then, $L^p \subset M_{p,q}^s$ if and only if one of the following conditions holds:

- (1) $p > 1$, $s \leq A(p, q)$ with strict inequality when $1/q > 1/p$,
- (2) $p = 1$, $s \leq A(1, q)$ with strict inequality when $q \neq \infty$.

Theorem D (cf. [13]) Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. Then, $M_{p,q}^s \subset L^p$ if and only if one of the following conditions holds:

- (1) $p < \infty$, $s \geq B(p, q)$ with strict inequality when $1/p > 1/q$,
- (2) $p = \infty$, $s \geq B(p, q)$ with strict inequality when $q > 1$.

As mentioned before, α -modulation space is a generalization of modulation space, and that can not be obtained by interpolation between modulation and Besov spaces, so it is interesting to ask what is the sharp conditions for the embedding between α -modulation and local Hardy spaces (Sobolev spaces). The main purpose of this paper is to give the answer. Now, we state our main theorems as follows.

Theorem 1.1. Let $\alpha \in [0, 1)$, $0 < p_1 < \infty$, $0 < p_2, q_2 \leq \infty$, $s_2 \in \mathbb{R}$. Then

$$h^{p_1} \subset M_{p_2, q_2}^{s_2, \alpha} \quad (1.1)$$

if and only if $1/p_2 \leq 1/p_1$ and

$$s_2 \leq n\alpha(1/p_2 - 1/p_1) + (1 - \alpha)A(p_1, q_2), \quad (1.2)$$

with strict inequality in (1.2) when $1/q_2 > 1/p_1$.

Theorem 1.2. Let $\alpha \in [0, 1)$, $0 < p_2 < \infty$, $0 < p_1, q_1 \leq \infty$, $s_1 \in \mathbb{R}$. Then

$$M_{p_1, q_1}^{s_1, \alpha} \subset h^{p_2} \quad (1.3)$$

if and only if $1/p_2 \leq 1/p_1$ and

$$s_1 \geq n\alpha(1/p_1 - 1/p_2) + (1 - \alpha)B(p_2, q_1), \quad (1.4)$$

with strict inequality in (1.4) when $1/p_2 > 1/q_1$.

Theorem 1.3. Let $\alpha \in [0, 1)$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$. Then

$$L^1 \subset M_{p, q}^{s, \alpha} \quad (1.5)$$

if and only if $1/p \leq 1$ and

$$s \leq n\alpha(1/p - 1) + n(1 - \alpha)A(1, q) \quad (1.6)$$

with the strict inequality in (1.6) when $q < \infty$.

Theorem 1.4. *Let $\alpha \in [0, 1)$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$. Then*

$$M_{p,q}^{s,\alpha} \subset L^1 \quad (1.7)$$

if and only if $1/p \geq 1$, and

$$s \geq n\alpha(1/p - 1) + (1 - \alpha)B(1, q) \quad (1.8)$$

with the strict inequality in (1.8) when $q > 1$.

Theorem 1.5. *Let $\alpha \in [0, 1)$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$. Then*

$$L^\infty \subset M_{p,q}^{s,\alpha} \quad (1.9)$$

if and only if $p = \infty$ and

$$s \leq n(1 - \alpha)A(\infty, q) \quad (1.10)$$

with the strict inequality in (1.10) when $q < \infty$.

Theorem 1.6. *Let $\alpha \in [0, 1)$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$. Then*

$$M_{p,q}^{s,\alpha} \subset L^\infty \quad (1.11)$$

if and only if

$$s \geq n\alpha/p + n(1 - \alpha)B(\infty, q),$$

with strict inequality when $q > 1$.

Remark 1.7. Obviously, our theorems generalize the main results in [12] and [13]. If we choose $p_1 = p_2$ and $\alpha = 0$ in our theorems, we regain the results obtained in [12, 13]. Furthermore, we use a quite different method to achieve our goals. For the necessity part, we reduce the embedding between local Hardy and α -modulation spaces to the corresponding embedding associated with discrete sequences. For the sufficiency, we estimate the h^p atoms under the "standard" norm of α -modulation spaces. Our method also works well in the special case $\alpha = 0$. Without using an equivalent norm of modulation space (see Lemma 2.2 in [12]), our method seems more readable and efficient. We also remark that a similar embedding problem (the case $p_1 = p_2$) is studied by T. Kato in his doctoral thesis [11], in which he use the same method as in [12]. Due to the generality of p_1, p_2 and the quite different method, our work has its own interesting.

Remark 1.8. Recalling $h^p \sim L^p$ for $p \in (1, \infty)$. Except for some endpoints, it is convenient to deal with the case $p \leq 1$ and $p \geq 1$ together. Thus, we establish Theorem 1.1 and Theorem 1.2 for all the embedding between α -modulation and local Hardy spaces in the full range. Then, we use Theorem 1.3 to 1.6 to deal with the endpoint cases.

Remark 1.9. As mentioned before, Besov space can be regarded as the special α -modulation space with $\alpha = 1$. However, the embedding relations with $\alpha = 1$ are the special cases of embedding between Besov and Triebel-Lizorkin spaces, which have been fully studied. So we focus on $\alpha \in [0, 1)$ in this paper.

Our paper is organized as follows. In Section 2, we give some definitions of function spaces treated in this paper. We also collect some basic properties which is useful in our proof. Section 3 is prepared for some estimates of h^p -atoms, which lead to the key embedding relations of our theorems. The proofs of our main results will be given in Section 4 to 7.

2. PRELIMINARIES AND DEFINITIONS

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. Let $\mathbb{N}_0 := \mathbb{Z}^+ \cup \{0\}$ be the collection of all nonnegative integers. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \mathcal{F}^{-1}f(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

The notation $X \lesssim Y$ denotes the statement that $X \leq CY$, with a positive constant C that may depend on $n, \alpha, p_i, q_i, s_i (i = 1, 2)$, but it might be different from line to line. The notation $X \sim Y$

means the statement $X \lesssim Y \lesssim X$. And the notation $X \simeq Y$ stands for $X = CY$. For a multi-index $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, we denote $|k|_\infty := \max_{i=1,2,\dots,n} |k_i|$ and $\langle k \rangle := (1 + |k|^2)^{1/2}$.

We recall some definitions of the function spaces treated in this paper.

First we give the partition of unity on frequency space associated with $\alpha \in [0, 1)$. Take two appropriate constants $c > 0$ and $C > 0$ and choose a Schwartz function sequence $\{\eta_k^\alpha\}_{k \in \mathbb{Z}^n}$ satisfying

$$\begin{cases} |\eta_k^\alpha(\xi)| \geq 1, & \text{if } |\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k| < c \langle k \rangle^{\frac{\alpha}{1-\alpha}}; \\ \text{supp} \eta_k^\alpha \subset \{\xi \in \mathbb{R}^n : |\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k| < C \langle k \rangle^{\frac{\alpha}{1-\alpha}}\}; \\ \sum_{k \in \mathbb{Z}^n} \eta_k^\alpha(\xi) \equiv 1, \forall \xi \in \mathbb{R}^n; \\ |\partial^\gamma \eta_k^\alpha(\xi)| \leq C_{|\alpha|} \langle k \rangle^{-\frac{\alpha|\gamma|}{1-\alpha}}, \forall \xi \in \mathbb{R}^n, \gamma \in (\mathbb{Z}^+ \cup \{0\})^n. \end{cases} \quad (2.1)$$

The sequence $\{\eta_k^\alpha\}_{k \in \mathbb{Z}^n}$ constitutes a smooth decomposition of \mathbb{R}^n . The frequency decomposition operators associated with the above function sequence are defined by

$$\square_k^\alpha := \mathcal{F}^{-1} \eta_k^\alpha \mathcal{F} \quad (2.2)$$

for $k \in \mathbb{Z}^n$. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $\alpha \in [0, 1)$. Then the α -modulation space associated with above decomposition is defined by

$$M_{p,q}^{s,\alpha}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{s q}{1-\alpha}} \|\square_k^\alpha f\|_{L^p}^q \right)^{1/q} < \infty\} \quad (2.3)$$

with the usual modifications when $q = \infty$. For simplicity, we write $M_{p,q}^s = M_{p,q}^{s,0}$, $M_{p,q} = M_{p,q}^{0,0}$ and $\eta_k(\xi) = \eta_k^0(\xi)$.

Remark 2.1. We recall that the above definition is independent of the choice of exact η_k^α (see [10]). Also, for sufficiently small $\delta > 0$, one can construct a function sequence $\{\eta_k^\alpha\}_{k \in \mathbb{Z}^n}$ such that $\eta_k^\alpha(\xi) = 1$, and $\eta_k^\alpha(\xi) \eta_l^\alpha(\xi) = 0$ if $k \neq l$ and ξ lies in the ball $B(\langle k \rangle^{\frac{\alpha}{1-\alpha}} k, \langle k \rangle^{\frac{\alpha}{1-\alpha}} \delta)$ (see [3, 7]).

Lemma 2.2 (Embedding of α -modulation spaces). *Let $0 < p_1, p_2, q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$, $\alpha \in [0, 1)$. Then $M_{p_1,q_1}^{s_1,\alpha} \subset M_{p_2,q_2}^{s_2,\alpha}$ if and only if*

$$\begin{cases} \frac{1}{p_2} \leq \frac{1}{p_1} \\ \frac{1}{q_2} \leq \frac{1}{q_1} \\ \frac{s_2}{n} - \frac{\alpha}{p_2} \leq \frac{s_1}{n} - \frac{\alpha}{p_1} \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{p_2} \leq \frac{1}{p_1} \\ \frac{1}{q_2} > \frac{1}{q_1} \\ \frac{s_2}{n} - \frac{\alpha}{p_2} + \frac{1-\alpha}{q_2} < \frac{s_1}{n} - \frac{\alpha}{p_1} + \frac{1-\alpha}{q_1}. \end{cases} \quad (2.4)$$

The above embedding lemma can be verified by the same method used in the proof of embedding between modulation spaces. We omit the details here. One can see [9] for a more general case. Next, we give some definitions and properties of sequences associated with α .

Definition 2.3. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $\alpha \in [0, 1)$. Let $\vec{a} := \{a_k\}_{k \in \mathbb{Z}^n}$ be a sequence, we denote its $l_p^{s,\alpha}$ (quasi-)norm

$$\|\vec{a}\|_{l_p^{s,\alpha}} = \begin{cases} \left(\sum_{k \in \mathbb{Z}^n} |a_k|^p \langle k \rangle^{\frac{s p}{1-\alpha}} \right)^{1/p}, & p < \infty \\ \sup_{k \in \mathbb{Z}^n} |a_k| \langle k \rangle^{\frac{s}{1-\alpha}}, & p = \infty \end{cases} \quad (2.5)$$

$$\quad (2.6)$$

and let $l_p^{s,\alpha}$ be the (quasi-)Banach space of sequences whose $l_p^{s,\alpha}$ (quasi-)norm is finite. Let $\vec{b} := \{b_j\}_{j \in \mathbb{N}}$ be a sequence, we denote its $l_p^{s,1}$ (quasi-)norm

$$\|\vec{b}\|_{l_p^{s,1}} = \begin{cases} \left(\sum_{j \in \mathbb{N}_0} |b_j|^p 2^{j s p} \right)^{1/p}, & p < \infty \\ \sup_{j \in \mathbb{N}_0} |b_j| 2^{j s}, & p = \infty \end{cases} \quad (2.7)$$

$$\quad (2.8)$$

and let $l_p^{s,1}$ be the (quasi-)Banach space of sequences whose $l_p^{s,1}$ (quasi-)norm is finite.

We also give the following lemma for the embedding about sequences defined above. The proof is quite standard, we omit the details here.

Lemma 2.4 (Sharpness of embedding, for α -decomposition). *Suppose $0 < q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$, $\alpha \in [0, 1]$. Then*

$$l_{q_1}^{s_1, \alpha} \subset l_{q_2}^{s_2, \alpha} \quad (2.9)$$

holds if and only if

$$(1 - \alpha)/q_2 + s_2/n < (1 - \alpha)/q_1 + s_1/n \quad \text{or} \quad \begin{cases} s_2 = s_1, \\ 1/q_2 \leq 1/q_1. \end{cases} \quad (2.10)$$

To define the Besov spaces and Triebel-Lizorkin spaces, we introduce the dyadic decomposition of \mathbb{R}^n . Let φ be a smooth bump function supported in the ball $\{\xi : |\xi| < \frac{3}{2}\}$ and be equal to 1 on the ball $\{\xi : |\xi| \leq \frac{4}{3}\}$. Denote

$$\phi(\xi) = \varphi(\xi) - \varphi(2\xi), \quad (2.11)$$

and a function sequence

$$\begin{cases} \phi_j(\xi) = \phi(2^{-j}\xi), \quad j \in \mathbb{Z}^+, \\ \phi_0(\xi) = 1 - \sum_{j \in \mathbb{Z}^+} \phi_j(\xi) = \varphi(\xi). \end{cases} \quad (2.12)$$

For integers $j \in \mathbb{N}_0$, we define the Littlewood-Paley operators

$$\Delta_j = \mathcal{F}^{-1} \phi_j \mathcal{F}. \quad (2.13)$$

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$. For a tempered distribution f , we set the norm

$$\|f\|_{B_{p,q}^s} = \left(\sum_{j \in \mathbb{N}_0} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{1/q}, \quad (2.14)$$

with the usual modifications when $q = \infty$. The (inhomogeneous) Besov space $B_{p,q}^s$ is the space of all tempered distributions f for which the quantity $\|f\|_{B_{p,q}^s}$ is finite. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. For a tempered distribution f , we set the norm

$$\|f\|_{F_{p,q}^s} = \left\| \left(\sum_{j \in \mathbb{N}_0} 2^{jsq} |\Delta_j f|^q \right)^{1/q} \right\|_{L^p} \quad (2.15)$$

with the usual modifications when $q = \infty$. The Triebel-Lizorkin space $F_{p,q}^s$ is the space of all tempered distributions f for which the quantity $\|f\|_{F_{p,q}^s}$ is finite.

Now, we turn to introduce the local Hardy space of Goldberg [4]. Let $\psi \in \mathcal{S}$ satisfy $\int_{\mathbb{R}^n} \psi(x) dx \neq 0$. Define $\psi_t = t^{-n} \psi(x/t)$. The *local Hardy spaces* is defined by

$$h^p := \{f \in \mathcal{S}' : \|f\|_{h^p} = \sup_{0 < t < 1} \|\psi_t * f\|_{L^p} < \infty\}.$$

We note that the definition of the local Hardy spaces is independent of the choice of $\psi \in \mathcal{S}$. A function a is said to be an h^p -atom of type I (small h^p -atom) if it satisfies the following *support condition*, *size condition* and *vanishing moment condition*:

$$\begin{aligned} & \text{supp } a \subset Q \text{ with } |Q| < 1, \|a\|_{L^\infty} \leq |Q|^{-1/p}, \\ & \int x^\beta a(x) dx = 0 \text{ for all } |\beta| \leq [n(1/p - 1)], \end{aligned} \quad (2.16)$$

where Q is a cube and $|Q|$ is the Lebesgue measure of Q , and $[n(1/p - 1)]$ is the integer part of $n(1/p - 1)$. A function a is said to be an h^p -atom of type II (big h^p -atom) if it satisfies

$$\text{supp } a \subset Q \text{ with } |Q| \geq 1, \|a\|_{L^\infty} \leq |Q|^{-1/p}. \quad (2.17)$$

All the big and small h_p -atoms are collectively called h_p -atom. We recall $\|a\|_{h_p} \lesssim 1$ for all h_p -atoms. On the other hand, any $f \in h_p$ ($p \leq 1$) can be represented by

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \quad (2.18)$$

where the series converges in the sense of distribution, $\{a_j\}$ is a collection of h_p -atoms and $\{\lambda_j\}$ is a sequence of complex numbers such that $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, we have

$$\|f\|_{h^p} \sim \inf \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p}, \quad (2.19)$$

where the infimum is taken over all representations $f = \sum_{j=1}^{\infty} \lambda_j a_j$ (see Lemma 5 in [4]).

We recall that the local Hardy space h^p is equivalent with the inhomogeneous Triebel-Lizorkin space $F_{p,2}^0$ for $p \in (0, \infty)$ (see 1.4 in [18]). For the sake of convenience, we usually use $F_{p,2}^0$ as the norm of local Hardy space throughout the remainder of this article.

Lemma 2.5 (Young's inequality).

(1) Let $0 < p \leq 1$, $R > 0$, $\text{supp} \hat{f}, \text{supp} \hat{g} \subseteq B(x, R) \subseteq \mathbb{R}^n$. We have

$$\|f * g\|_{L^p} \leq C R^{n(\frac{1}{p}-1)} \|f\|_{L^p} \|g\|_{L^p}, \quad (2.20)$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $R > 0$, where C is independent of x , $x \in \mathbb{R}^n$.

(2) Let $1 \leq p, q, r \leq \infty$ satisfy $1 + 1/q = 1/p + 1/r$. Then we have

$$\|f * g\|_{L^q} \lesssim \|f\|_{L^p} \|g\|_{L^r}. \quad (2.21)$$

The following proposition shows that the local Hardy space h^p is equivalent with L^p in the local meaning.

Proposition 2.6. Let $0 < p < \infty$, $\alpha \in [0, 1)$. Suppose $\{\varphi_k\}_{k \in \mathbb{Z}^n}$ is a sequence of smooth functions such that

$$\text{supp} \varphi_k \subset B(\langle k \rangle^{\frac{\alpha}{1-\alpha}}, C \langle k \rangle^{\frac{\alpha}{1-\alpha}})$$

where C is a fixed positive constant. Then we have

$$\|\varphi_k\|_{h^p} \sim \|\varphi_k\|_{L^p}$$

for all $k \in \mathbb{Z}^n$.

Proof. By the definition of local Hardy space, we have $\|\varphi_k\|_{h^p} \gtrsim \|\varphi_k\|_{L^p}$. For the opposite direction, we use the Littlewood-Paley characterization of the local Hardy space. By Lemma 2.5, we have

$$\|\varphi_k\|_{h^p} = \|\varphi_k\|_{F_{p,2}^0} = \left\| \left(\sum_{j \in \mathbb{Z}^n} |\Delta_j \varphi_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \sum_{j \in \mathbb{Z}^n} \|\Delta_j \varphi_k\|_{L^p} \lesssim \|\varphi_k\|_{L^p}, \quad (2.22)$$

where we use the fact that there are only finite j such that $\Delta_j \varphi_k$ is not zero. \square

3. ESTIMATES OF h^p ATOMS ON α -MODULATION SPACES

In order to make the proof of main theorems more clear, we give some important estimates associated with h^p -atoms.

Proposition 3.1 (Pointwise estimate for h^p atoms). *Suppose a is a function supported in a cube Q centered at the origin, satisfying $\|a\|_{L^\infty} \leq |Q|^{-1/p}$. We let Q^* be the cube with side length $2\sqrt{n}l(Q)$ having the same center as Q , where the $l(Q)$ is the side length of Q . Take ρ to be a smooth function with compact support near the origin. Denote $\rho_k^\alpha(\xi) = \rho\left(\frac{\xi - \langle k \rangle^{\frac{1-\alpha}{1-\alpha}} k}{\langle k \rangle^{\frac{1-\alpha}{1-\alpha}}}\right)$ and $\tilde{\square}_k^\alpha = \rho_k^\alpha(D)$. Then we have*

$$|\tilde{\square}_k^\alpha a(x)| \lesssim \langle k \rangle^{\frac{n\alpha}{1-\alpha}} \langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}} x \rangle^{-\mathcal{L}} |Q|^{1-1/p} \text{ for } x \in (Q^*)^c,$$

where \mathcal{L} is a fixed number which could be arbitrary large. If the function a also satisfies the vanishing moment condition as follow

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0 \text{ for all } |\beta| \leq N := [n(1/p - 1)],$$

we also have the following estimate

$$|\tilde{\square}_k^\alpha a(x)| \lesssim \langle k \rangle^{\frac{n\alpha}{1-\alpha}} (\sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}})^{N+1} \langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}} x \rangle^{-\mathcal{L}} |Q|^{1-1/p}, \text{ for } x \in (Q^*)^c,$$

where \mathcal{L} is a fixed number which could be arbitrary large.

Proof. Using the assumption, we obtain $\|a\|_{L^1} \leq \|a\|_{L^\infty} \cdot |Q| \leq |Q|^{1-1/p}$. By the rapid decay of $\mathcal{F}^{-1}\rho$, we deduce that

$$\begin{aligned} |\tilde{\square}_k^\alpha a(x)| &= \left| \int_{\mathbb{R}^n} (\rho_k^\alpha)^\vee(x-y) a(y) dy \right| = \left| \int_Q (\rho_k^\alpha)^\vee(x-y) a(y) dy \right| \\ &\leq \int_Q |\langle k \rangle^{\frac{n\alpha}{1-\alpha}} \rho^\vee(\langle k \rangle^{\frac{n\alpha}{1-\alpha}}(x-y)) a(y)| dy \\ &\lesssim \langle k \rangle^{\frac{n\alpha}{1-\alpha}} \int_Q |\langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}}(x-y) \rangle^{-\mathcal{L}} a(y)| dy \\ &\lesssim \langle k \rangle^{\frac{n\alpha}{1-\alpha}} \langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}} x \rangle^{-\mathcal{L}} \|a\|_{L^1} \\ &\lesssim \langle k \rangle^{\frac{n\alpha}{1-\alpha}} \langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}} x \rangle^{-\mathcal{L}} |Q|^{1-1/p}. \end{aligned} \tag{3.1}$$

In addition, if a satisfies the vanishing moment condition, we use the Taylor formula to deduce that

$$\begin{aligned} \tilde{\square}_k^\alpha a(x) &= \int_{\mathbb{R}^n} (\rho_k^\alpha)^\vee(x-y) a(y) dy \\ &= \int_{\mathbb{R}^n} \left((\rho_k^\alpha)^\vee(x-y) - \sum_{|\gamma| \leq N} \frac{\partial^\gamma (\rho_k^\alpha)^\vee(x)}{|\gamma|!} (-y)^\gamma \right) a(y) dy \\ &= \int_{\mathbb{R}^n} \sum_{|\gamma| = N+1} \int_0^1 \frac{(-y)^\gamma}{N!} (1-t)^N \partial^\gamma (\rho_k^\alpha)^\vee(x-ty) a(y) dt dy. \end{aligned} \tag{3.2}$$

Noticing that $(\rho_k^\alpha)^\vee(x) = \langle k \rangle^{\frac{n\alpha}{1-\alpha}} \rho^\vee(\langle k \rangle^{\frac{n\alpha}{1-\alpha}} x) e^{2\pi i \langle k \rangle^{\frac{n\alpha}{1-\alpha}} k \cdot x}$, we obtain that

$$\begin{aligned} |(\partial^\gamma (\rho_k^\alpha)^\vee)(x-ty)| &\lesssim \langle k \rangle^{\frac{n\alpha}{1-\alpha}} \sum_{\gamma_1 + \gamma_2 = \gamma} \langle k \rangle^{\frac{\alpha|\gamma_1|}{1-\alpha}} \langle k \rangle^{\frac{|\gamma_2|}{1-\alpha}} \langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}}(x-ty) \rangle^{-\mathcal{L}} \\ &\lesssim \langle k \rangle^{\frac{n\alpha}{1-\alpha}} \langle k \rangle^{\frac{|\gamma|}{1-\alpha}} \langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}} x \rangle^{-\mathcal{L}} \end{aligned} \tag{3.3}$$

for $x \in (Q^*)^c$, $y \in Q$, $t \in [0, 1]$. Combining with (3.2) and (3.3), we deduce that

$$\begin{aligned} |\tilde{\square}_k^\alpha a(x)| &\lesssim \int_Q \sum_{|\gamma| = N+1} \int_0^1 \frac{|y|^{N+1}}{N!} (1-t)^N |(\partial^\gamma (\rho_k^\alpha)^\vee)(x-ty)| \cdot |a(y)| dt dy \\ &\lesssim \langle k \rangle^{\frac{n\alpha}{1-\alpha}} \langle k \rangle^{\frac{N+1}{1-\alpha}} \langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}} x \rangle^{-\mathcal{L}} \int_Q |y|^{N+1} |a(y)| dy \\ &\lesssim \langle k \rangle^{\frac{n\alpha}{1-\alpha}} \langle k \rangle^{\frac{N+1}{1-\alpha}} \langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}} x \rangle^{-\mathcal{L}} |Q|^{\frac{N+1}{n}} |Q|^{1-1/p} \\ &= \langle k \rangle^{\frac{n\alpha}{1-\alpha}} (\sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}})^{N+1} \langle \langle k \rangle^{\frac{n\alpha}{1-\alpha}} x \rangle^{-\mathcal{L}} |Q|^{1-1/p} \end{aligned}$$

for $x \in (Q^*)^c$. □

Proposition 3.2. *Let $0 < p \leq 1$, we have*

$$\|a\|_{M_{p,\infty}^{n(1-\alpha)(1-1/p),\alpha}} \lesssim 1$$

for all h^p -atoms.

Proof. Take a to be an h^p atom, and without loss of generality we may assume that a is supported in a cube Q centered at the origin. We denote Q^* the cube with side length $2\sqrt{n}l(Q)$ having the same center as Q , where the $l(Q)$ is the side length of Q . Choose a smooth function ρ with compact support, such that $\rho_k^\alpha \cdot \eta_k^\alpha = \eta_k^\alpha$ and $\square_k^\alpha \circ \tilde{\square}_k^\alpha = \square_k^\alpha$, where we denote $\rho_k^\alpha(\xi) = \rho\left(\frac{\xi - \langle k \rangle^{\frac{1-\alpha}{1-\alpha}} k}{\langle k \rangle^{\frac{1-\alpha}{1-\alpha}}}\right)$ and $\tilde{\square}_k^\alpha = \rho_k^\alpha(D)$. Using Lemma 2.5, we obtain

$$\|\square_k^\alpha f\|_{L^p} = \|\tilde{\square}_k^\alpha(\square_k^\alpha f)\|_{L^p} \lesssim \langle k \rangle^{\frac{n\alpha(1/(p\wedge 1)-1)}{1-\alpha}} \|\mathcal{F}^{-1}\eta_k^\alpha\|_{L^{p\wedge 1}} \cdot \|\tilde{\square}_k^\alpha f\|_{L^p} \lesssim \|\tilde{\square}_k^\alpha f\|_{L^p} \quad (3.4)$$

for any fixed $p \in (0, \infty]$. Thus, we only need to verify

$$\|a\|_{\tilde{M}_{p,\infty}^{n(1-\alpha)(1-1/p),\alpha}} := \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a\|_{L^p} \lesssim 1.$$

By the (quasi-)triangle inequality, we have

$$\begin{aligned} \|a\|_{\tilde{M}_{p,\infty}^{n(1-\alpha)(1-1/p),\alpha}} &= \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a\|_{L^p} \\ &\lesssim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a \chi_{Q^*}\|_{L^p} + \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p} \end{aligned} \quad (3.5)$$

By the properties of h^p -atom, we obtain

$$\|\tilde{\square}_k^\alpha a \chi_{Q^*}\|_{L^p} \lesssim |Q|^{1/p} \|\tilde{\square}_k^\alpha a\|_{L^\infty} \lesssim |Q|^{1/p} \|a\|_{L^\infty} \lesssim 1.$$

Recalling $p \leq 1$, we deduce $\sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a \chi_{Q^*}\|_{L^p} \lesssim 1$.

We turn to the estimates of second term in (3.5). Denote $N = [n(1/p - 1)]$. Using Proposition 3.1, we obtain that

$$\begin{aligned} &\sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p} \\ &\lesssim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\langle k \rangle^{\frac{n\alpha}{1-\alpha}} (\sqrt[p]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}})^{N+1} \langle k \rangle^{\frac{\alpha}{1-\alpha}} x^{-\mathcal{L}} |Q|^{1-1/p} \chi_{(Q^*)^c}\|_{L^p} \\ &\lesssim (\sqrt[p]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}})^{n(1-1/p)+N+1} \leq 1, \end{aligned} \quad (3.6)$$

where we use $N+1+n(1-1/p) \geq 0$ and the fact that there is no $k \in \mathbb{Z}^n$ such that $\sqrt[p]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} < 1$ for big h_p -atom. We also have

$$\begin{aligned} &\sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p} \\ &\lesssim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\langle k \rangle^{\frac{n\alpha}{1-\alpha}} \langle k \rangle^{\frac{\alpha}{1-\alpha}} x^{-\mathcal{L}} |Q|^{1-1/p} \chi_{(Q^*)^c}\|_{L^p} \\ &\lesssim \sup_{k \in \mathbb{Z}^n} (\sqrt[p]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}})^{n(1-1/p)} \leq 1, \end{aligned}$$

where we use $n(1 - 1/p) \leq 0$. Thus, we deduce that

$$\begin{aligned} & \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p} \\ & \lesssim \sup_{\substack{k \in \mathbb{Z}^n \\ \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} < 1}} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p} + \sup_{\substack{k \in \mathbb{Z}^n \\ \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} \geq 1}} \langle k \rangle^{n(1-1/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p} \lesssim 1. \end{aligned}$$

□

Proposition 3.3. *Let $0 < p < 1$, we have*

$$\|a\|_{M_{p,p}^{n(1-\alpha)(1-2/p), \alpha}} \lesssim 1$$

for all h^p -atoms.

Proof. Take a to be an h^p atom, and without loss of generality we may assume that a is supported in a cube Q centered at the origin. We denote Q^* the cube with side length $2\sqrt{n}l(Q)$ having the same center as Q , where the $l(Q)$ is the side length of Q . By the same argument as in the proof of Proposition 3.2, we only need to verify

$$\|a\|_{\widetilde{M}_{p,p}^{n(1-\alpha)(1-2/p), \alpha}} := \|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} \lesssim 1.$$

By the (quasi-)triangle inequality, we have

$$\begin{aligned} \|a\|_{\widetilde{M}_{p,p}^{n(1-\alpha)(1-2/p), \alpha}} &= \|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} \\ &\lesssim \|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a \chi_{Q^*}\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} + \|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} \end{aligned}$$

Recalling $\|\tilde{\square}_k^\alpha a \chi_{Q^*}\|_{L^p} \lesssim 1$ obtained in the proof of Proposition 3.2, we deduce that

$$\|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a \chi_{Q^*}\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} \lesssim \|\{\langle k \rangle^{n(1-2/p)}\}_{k \in \mathbb{Z}^n}\|_{l^p} \lesssim 1, \quad (3.7)$$

where we use the fact $p < 1$. On the other hand, by the same method as in the proof of Proposition 3.2, we obtain that

$$\begin{aligned} & \|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} < 1 \|_{l_k^p} \\ & \lesssim \|\{\langle k \rangle^{-n/p} (\sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}})^{n(1-1/p)+N+1}\}_{k \in \mathbb{Z}^n}\|_{l_k^p} \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} < 1 \|_{l_k^p} \\ & = (\sqrt[n]{|Q|})^{n(1-1/p)+N+1} \left(\sum_{\substack{k \in \mathbb{Z}^n \\ \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} < 1}} \langle k \rangle^{-n} (\langle k \rangle^{\frac{p}{1-\alpha}})^{n(1-1/p)+N+1} \right)^{1/p} \lesssim 1. \end{aligned} \quad (3.8)$$

where we use the fact $N + 1 + n(1 - 1/p) > 0$. We also have

$$\begin{aligned} & \|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} \geq 1 \|_{l_k^p} \\ & \lesssim \|\{\langle k \rangle^{-n/p} (\sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}})^{n(1-1/p)}\}_{k \in \mathbb{Z}^n}\|_{l_k^p} \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} \geq 1 \|_{l_k^p} \\ & = (\sqrt[n]{|Q|})^{n(1-1/p)} \left(\sum_{\substack{k \in \mathbb{Z}^n \\ \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} \geq 1}} \langle k \rangle^{-n} (\langle k \rangle^{\frac{p}{1-\alpha}})^{n(1-1/p)} \right)^{1/p} \lesssim 1, \end{aligned} \quad (3.9)$$

where we use the fact $p < 1$. Thus,

$$\begin{aligned} \|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} &\lesssim \|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} < 1 \|_{l_k^p} \\ &\quad + \|\{\langle k \rangle^{n(1-2/p)} \|\tilde{\square}_k^\alpha a \chi_{(Q^*)^c}\|_{L^p}\}_{k \in \mathbb{Z}^n}\|_{l^p} \sqrt[n]{|Q|} \langle k \rangle^{\frac{1}{1-\alpha}} \geq 1 \|_{l_k^p} \lesssim 1. \end{aligned} \quad (3.10)$$

□

4. PROOF OF THEOREM 1.1

In this section, we give the proof of Theorem 1.1, i.e. the sharp conditions of $h^{p_1} \subset M_{p_2, q_2}^{s_2, \alpha}$.

4.1. Necessity of Theorem 1.1. To verify the necessity, we first show that the embedding $h^{p_1} \subset M_{p_2, q_2}^{s_2, \alpha}$ actually implies the embedding about corresponding weighted sequences.

Proposition 4.1. *Let $0 < p_1 < \infty$, $0 < p_2, q_2 \leq \infty$, $s \in \mathbb{R}$. If $h^{p_1} \subset M_{p_2, q_2}^{s_2, \alpha}$, then we have*

(1)

$$1/p_2 \leq 1/p_1,$$

(2)

$$l_{p_1}^{n(1-1/p_1), 1} \subset l_{q_2}^{n(1-\alpha)/q_2 + n\alpha(1-1/p_2) + s_2, 1},$$

(3)

$$l_{p_1}^{n\alpha(1-1/p_1), \alpha} \subset l_{q_2}^{n\alpha(1-1/p_2) + s_2, \alpha}.$$

Proof. We only state the proof for $q_2 < \infty$, since the case of $q_2 = \infty$ can be treated by the same method with a slight modification. By the Littlewood-Paley characterization of local Hardy space, we actually have

$$\|f\|_{M_{p_2, q_2}^{s_2, \alpha}} \lesssim \|f\|_{F_{p_1, 2}^0}. \quad (4.1)$$

Let $f \in \mathcal{S}$ be a nonzero function whose Fourier transform has compact support in $B(0, 1)$. Set $\widehat{f_\lambda}(\xi) = \widehat{f}(\xi/\lambda)$, $\lambda > 0$. Using (4.1) and the local property of α -modulation and local Hardy space, we obtain $\|f_\lambda\|_{L^{p_2}} \lesssim \|f_\lambda\|_{L^{p_1}}$ for sufficiently small λ and consequently $\lambda^{n(1-1/p_2)} \lesssim \lambda^{n(1-1/p_1)}$. Letting $\lambda \rightarrow 0^+$, we deduce

$$1/p_2 \leq 1/p_1, \quad (4.2)$$

which is the first desired condition.

Let g be a nonzero Schwartz function whose Fourier transform has compact support in $\{\xi : 3/4 \leq |\xi| \leq 4/3\}$, satisfying $g(\xi) = 1$ on $\{\xi : 7/8 \leq |\xi| \leq 8/7\}$. Set $\widehat{g_j}(\xi) := \widehat{g}(\xi/2^j)$. By the definition of Δ_j , we have $\Delta_j g_j = g_j$ for $j \geq 0$, and $\Delta_l g_j = 0$ if $l \neq j$. Denote

$$\widetilde{\Gamma_j} = \{k \in \mathbb{Z}^n : \text{supp} \eta_k^\alpha \subset \{\xi : (7/8) \cdot 2^j \leq |\xi| \leq (8/7) \cdot 2^j\}\}, \quad (4.3)$$

we have $|\widetilde{\Gamma_j}| \sim 2^{jn(1-\alpha)}$ for $j \geq N$, where N is a sufficiently large number. For a truncated (only finite nonzero items) nonnegative sequence $\vec{a} = \{a_j\}_{j \in \mathbb{N}}$, we define

$$G_N^h := \sum_{j \geq N} a_j g_j^h, \quad g_j^h(x) := g_j(x + jhe_0), \quad (4.4)$$

where $h \in \mathbb{R}$, $e_0 = (1, 0, \dots, 0)$ is the unit vector of \mathbb{R}^n . By the definition of α -modulation space, we obtain that

$$\begin{aligned} \|G_N^h\|_{M_{p_2, q_2}^{s_2, \alpha}} &= \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|\square_k^\alpha G_N^h\|_{L^{p_2}}^{q_2} \right)^{1/q_2} \geq \left(\sum_{j \in \mathbb{N}_0} \sum_{k \in \widetilde{\Gamma_j}} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|\square_k^\alpha G_N^h\|_{L^{p_2}}^{q_2} \right)^{1/q_2} \\ &= \left(\sum_{j=N}^{\infty} \sum_{k \in \widetilde{\Gamma_j}} a_j^{q_2} \|\mathcal{F}^{-1} \eta_k^\alpha\|_{L^{p_2}}^{q_2} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \right)^{1/q_2} \sim \left(\sum_{j=N}^{\infty} \sum_{k \in \widetilde{\Gamma_j}} a_j^{q_2} \langle k \rangle^{\frac{n\alpha q_2}{1-\alpha} (1-1/p_2) 2^{js_2 q_2}} \right)^{1/q_2} \\ &\sim \left(\sum_{j=N}^{\infty} \sum_{k \in \widetilde{\Gamma_j}} a_j^{q_2} 2^{jq_2(n\alpha(1-1/p_2) + s_2)} \right)^{1/q_2} \sim \left(\sum_{j=N}^{\infty} |\widetilde{\Gamma_j}| a_j^{q_2} 2^{jq_2(n\alpha(1-1/p_2) + s_2)} \right)^{1/q_2} \\ &= \left(\sum_{j=N}^{\infty} a_j^{q_2} 2^{jq_2(n\alpha(1-1/p_2) + s_2)} 2^{jn(1-\alpha)} \right)^{1/q_2} = \|\{a_j\}_{j \geq N}\|_{l_{q_2}^{n(1-\alpha)/q_2 + n\alpha(1-1/p_2) + s_2, 1}}. \end{aligned} \quad (4.5)$$

On the other hand, using the orthogonality of $\{g_j^h\}$ as $h \rightarrow \infty$, we deduce that

$$\begin{aligned}
\|G_N^h\|_{F_{p_1,2}^0} &= \left\| \left(\sum_{j \in \mathbb{N}_0} |\Delta_j G_N^h|^2 \right)^{1/2} \right\|_{L^{p_1}} = \left\| \left(\sum_{j=N}^{\infty} |a_j g_j^h|^2 \right)^{1/2} \right\|_{L^{p_1}} \\
&= \left(\int_{\mathbb{R}^n} \left(\sum_{j=N}^{\infty} |a_j g_j^h|^2 \right)^{p_1/2} dx \right)^{1/p_1} \xrightarrow{h \rightarrow \infty} \left(\int_{\mathbb{R}^n} \sum_{j=N}^{\infty} |a_j g_j|^{p_1} dx \right)^{1/p_1} \\
&\simeq \left(\sum_{j=N}^{\infty} a_j^{p_1} 2^{jn(1-1/p_1)p_1} \right)^{1/p_1} = \|\{a_j\}_{j \geq N}\|_{l_{p_1}^{n(1-1/p_1),1}}.
\end{aligned} \tag{4.6}$$

Combining (4.6) and (4.5), we have

$$\|\{a_j\}_{j \geq N}\|_{l_{q_2}^{n(1-\alpha)/q_2 + n\alpha(1-1/p_2) + s_2,1}} \lesssim \|\{a_j\}_{j \geq N}\|_{l_{p_1}^{n(1-1/p_1),1}},$$

which implies the desired embedding $l_{p_1}^{n(1-1/p_1),1} \subset l_{q_2}^{n(1-\alpha)/q_2 + n\alpha(1-1/p_2) + s_2,1}$.

Next, we turn to the proof of $l_{p_1}^{n\alpha(1-1/p_1),\alpha} \subset l_{q_2}^{n\alpha(1-1/p_2) + s_2,\alpha}$. Let $f \in \mathcal{S}$ be a nonzero smooth function whose Fourier transform has small support, such that $\square_k^\alpha f_k = f_k$ and $\square_l^\alpha f_k = 0$ if $k \neq l$, where we denote $\widehat{f_k}(x) = \widehat{f}\left(\frac{\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k}{\langle k \rangle^{\frac{\alpha}{1-\alpha}}}\right)$. For a truncated (only finite nonzero items) nonnegative sequence $\vec{b} = \{b_k\}_{k \in \mathbb{Z}^n}$, we define

$$F^h(x) = \sum_{k \in \mathbb{Z}^n} b_k f_k^h, \quad f_k^h(x) = f_k(x - kh), \tag{4.7}$$

where $h \in \mathbb{R}$. By a direct computation, we have

$$\begin{aligned}
\|F^h\|_{M_{p_2,q_2}^{s_2,\alpha}} &= \left(\sum_{k \in \mathbb{Z}^n} b_k^{q_2} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|f_k\|_{L^{p_2}}^{q_2} \right)^{1/q_2} \\
&\sim \left(\sum_{k \in \mathbb{Z}^n} b_k^{q_2} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \langle k \rangle^{\frac{n\alpha}{1-\alpha}(1-1/p_2)q_2} \right)^{1/q_2} \sim \|\vec{b}\|_{l_{q_2}^{n\alpha(1-1/p_2) + s_2,\alpha}}.
\end{aligned} \tag{4.8}$$

On the other hand, for $p_1 > 1$, using the orthogonality of $\{f_k^h\}_{k \in \mathbb{Z}^n}$ as $h \rightarrow \infty$, we obtain

$$\begin{aligned}
\|F^h\|_{h^{p_1}} &\sim \|F^h\|_{L^{p_1}} = \left(\int_{\mathbb{R}^n} \left| \sum_{k \in \mathbb{Z}^n} b_k f_k^h \right|^{p_1} dx \right)^{\frac{1}{p_1}} \xrightarrow{h \rightarrow \infty} \left(\int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} |b_k f_k|^{p_1} dx \right)^{\frac{1}{p_1}} \\
&\sim \left(\sum_{k \in \mathbb{Z}^n} b_k^{p_1} \langle k \rangle^{\frac{n\alpha}{1-\alpha}(1-1/p_1)p_1} \right)^{1/p_1} \sim \|\vec{b}\|_{l_{p_1}^{n\alpha(1-1/p_1),\alpha}}.
\end{aligned}$$

For $p_1 \leq 1$, we use quasi-triangle inequality to deduce that

$$\begin{aligned}
\|F^h\|_{h^{p_1}}^{p_1} &= \left\| \sum_{k \in \mathbb{Z}^n} b_k f_k^h \right\|_{h^{p_1}}^{p_1} \leq \sum_{k \in \mathbb{Z}^n} \|b_k f_k^h\|_{h^{p_1}}^{p_1} = \sum_{k \in \mathbb{Z}^n} \|b_k f_k\|_{L^{p_1}}^{p_1} \\
&\sim \sum_{k \in \mathbb{Z}^n} b_k^{p_1} \langle k \rangle^{\frac{n\alpha}{1-\alpha}(1-1/p_1)p_1} \sim \|\vec{b}\|_{l_{p_1}^{n\alpha(1-1/p_1),\alpha}}^{p_1},
\end{aligned}$$

where we use the fact $\|f_k\|_{L^{p_1}} \sim \|f_k\|_{L^{p_1}}$ (see Proposition 2.6). So we have

$$\lim_{h \rightarrow \infty} \|F^h\|_{h^{p_1}} \lesssim \|\vec{b}\|_{l_{p_1}^{n\alpha(1-1/p_1),\alpha}}^{p_1}. \tag{4.9}$$

Combining (4.8) and (4.9), we obtain

$$\|\vec{b}\|_{l_{q_2}^{n\alpha(1-1/p_2)+s_2,\alpha}} \lesssim \|\vec{b}\|_{l_{p_1}^{n\alpha(1-1/p_1),\alpha}} \quad (4.10)$$

which implies the desired embedding $l_{p_1}^{n\alpha(1-1/p_1),\alpha} \subset l_{q_2}^{n\alpha(1-1/p_2)+s_2,\alpha}$. \square

Now, we are in the position to verify the necessity for $h^{p_1} \subset M_{p_2,q_2}^{s_2,\alpha}$. By Proposition 4.1, we obtain $1/p_2 \leq 1/p_1$ and

$$l_{p_1}^{n(1-1/p_1),1} \subset l_{q_2}^{n(1-\alpha)/q_2+n\alpha(1-1/p_2)+s_2,1}, \quad l_{p_1}^{n\alpha(1-1/p_1),\alpha} \subset l_{q_2}^{n\alpha(1-1/p_2)+s_2,\alpha}.$$

By Lemma 2.4, $l_{p_1}^{n(1-1/p_1),1} \subset l_{q_2}^{n(1-\alpha)/q_2+n\alpha(1-1/p_2)+s_2,1}$ implies $s_2 \leq n\alpha(1/p_2 - 1/p_1) + n(1-\alpha)(1 - 1/p_1 - 1/q_2)$ and the inequality is strict if $1/q_2 > 1/p_1$. On the other hand, by Lemma 2.4 and $l_{p_1}^{n\alpha(1-1/p_1),\alpha} \subset l_{q_2}^{n\alpha(1-1/p_2)+s_2,\alpha}$, we obtain $s_2 \leq n\alpha(1/p_2 - 1/p_1)$ for $1/q_2 \leq 1/p_1$, and $s_2 < n(1-\alpha)(1/p_1 - 1/q_2) + n\alpha(1/p_2 - 1/p_1)$ for $1/q_2 > 1/p_1$.

Combining with the above estimates, we obtain $s_2 \leq n\alpha(1/p_2 - 1/p_1) + (1-\alpha)A(p_1, q_2)$, with strict inequality for $1/q_2 > 1/p_1$.

4.2. Sufficiency of Theorem 1.1. We only need to verify $h^{p_1} \subset M_{p_1,q_2}^{(1-\alpha)A(p_1,q_2),\alpha}$ for $1/q_2 \leq 1/p_1$, and $h^{p_1} \subset M_{p_1,q_2}^{(1-\alpha)A(p_1,q_2)-\epsilon,\alpha}$ for $1/q_2 > 1/p_1$, where ϵ is any positive number. Then the final conclusion follows by $M_{p_1,q_2}^{(1-\alpha)A(p_1,q_2),\alpha} \subset M_{p_2,q_2}^{n\alpha(1/p_2-1/p_1)+(1-\alpha)A(p_1,q_2),\alpha}$ or $M_{p_1,q_2}^{(1-\alpha)A(p_1,q_2)+\epsilon,\alpha} \subset M_{p_2,q_2}^{n\alpha(1/p_2-1/p_1)+(1-\alpha)A(p_1,q_2)+\epsilon,\alpha}$.

For $1/q_2 \leq 1/p_1$. We want to verify that

$$h^{p_1} \subset M_{p_1,q_2}^{(1-\alpha)A(p_1,q_2),\alpha}.$$

Taking a fixed $f \in h^{p_1}$ ($p_1 \leq 1$), we can find a collection of h_{p_1} -atoms $\{a_j\}_{j=1}^\infty$ and a sequence of complex numbers $\{\lambda_j\}_{j=1}^\infty$ which is depend on f , such that $f = \sum_{j=1}^\infty \lambda_j a_j$ and

$$\left(\sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{h_p}, \quad (4.11)$$

where the constant C is independent of f . For $p_1 \leq 1$, we use Proposition 3.2 to deduce that

$$\begin{aligned} \|f\|_{M_{p_1,\infty}^{(1-\alpha)A(p_1,\infty),\alpha}} &= \|f\|_{M_{p_1,\infty}^{n(1-\alpha)(1-1/p_1),\alpha}} = \left\| \sum_{j=1}^\infty \lambda_j a_j \right\|_{M_{p_1,\infty}^{n(1-\alpha)(1-1/p_1),\alpha}} \\ &\lesssim \left(\sum_{j=1}^\infty \|\lambda_j a_j\|_{M_{p_1,\infty}^{n(1-\alpha)(1-1/p_1),\alpha}}^{p_1} \right)^{1/p_1} \\ &\lesssim \left(\sum_{j=1}^\infty |\lambda_j|^{p_1} \right)^{1/p_1} \lesssim \|f\|_{h_{p_1}}. \end{aligned} \quad (4.12)$$

Similarly, we use Proposition 3.3 to deduce that

$$\|f\|_{M_{p_1,p_1}^{(1-\alpha)A(p_1,p_1),\alpha}} = \|f\|_{M_{p_1,p_1}^{n(1-\alpha)(1-2/p_1),\alpha}} \lesssim \|f\|_{h^{p_1}} \quad (4.13)$$

for $p_1 < 1$. By a direct calculation, we also have

$$\|f\|_{M_{\infty,\infty}^{(1-\alpha)A(\infty,\infty),\alpha}} = \|f\|_{M_{\infty,\infty}^{0,\alpha}} = \sup_{k \in \mathbb{Z}^n} \|\square_k^\alpha f\|_{L^\infty} \lesssim \|f\|_{L^\infty}. \quad (4.14)$$

Recalling $\|f\|_{M_{2,2}^{(1-\alpha)A(2,2),\alpha}} = \|f\|_{M_{2,2}^{0,\alpha}} \sim \|f\|_{L^2}$, we obtain the following inequality:

$$\|f\|_{M_{2,2}^{(1-\alpha)A(2,2),\alpha}} \lesssim \|f\|_{L^2}. \quad (4.15)$$

By an interpolation among (4.12), (4.13), (4.14) and (4.15), we obtain the desired conclusion.

For $1/q_2 > 1/p_1$. We want to verify that

$$h^{p_1} \subset M_{p_1, q_2}^{(1-\alpha)A(p_1, q_2) - \epsilon, \alpha},$$

where ϵ is any fixed positive number. In fact, observing that

$$((1-\alpha)A(p_1, q_2) - \epsilon)/n + (1-\alpha)/q_2 < ((1-\alpha)A(p_1, p_1)/n + (1-\alpha)/p_1$$

for $1/q_2 > 1/p_1$, we use Lemma 2.2 to deduce

$$M_{p_1, p_1}^{(1-\alpha)A(p_1, p_1), \alpha} \subset M_{p_1, q_2}^{(1-\alpha)A(p_1, q_2) - \epsilon, \alpha}.$$

Recalling $h^{p_1} \subset M_{p_1, p_1}^{(1-\alpha)A(p_1, p_1), \alpha}$, we obtain the desired conclusion:

$$h^{p_1} \subset M_{p_1, p_1}^{(1-\alpha)A(p_1, p_1), \alpha} \subset M_{p_1, q_2}^{(1-\alpha)A(p_1, q_2) - \epsilon, \alpha}$$

for any fixed positive number.

5. PROOF OF THEOREM 1.2

5.1. Necessity of Theorem 1.2. As in the proof of Theorem 1.1, we use following proposition to show that the necessity of $M_{p_2, q_2}^{s_2, \alpha} \subset h^{p_1}$ can be reduced to the necessity of embedding between corresponding weighted sequences.

Proposition 5.1. *Let $0 < p_2 < \infty$, $0 < p_1, q_1 \leq \infty$, $s_1 \in \mathbb{R}$. If $M_{p_1, q_1}^{s_1, \alpha} \subset h^{p_2}$, then we have*

(1)

$$1/p_2 \leq 1/p_1,$$

(2)

$$l_{q_1}^{n(1-\alpha)/q_1 + n\alpha(1-1/p_1) + s_1, 1} \subset l_{p_2}^{n(1-1/p_2), 1},$$

(3)

$$l_{q_1}^{n\alpha(1-1/p_1) + s_1, \alpha} \subset l_{p_2}^{n\alpha(1-1/p_2), \alpha}.$$

Proof. Take $f \in \mathcal{S}$ to be a nonzero smooth function whose Fourier transform has compact support in $B(0, 1)$. Denote $\widehat{f_\lambda}(\xi) = \widehat{f}(\xi/\lambda)$, $\lambda > 0$. By the assumption, we have

$$\|f_\lambda\|_{F_{p_2, 2}^0} \lesssim \|f_\lambda\|_{M_{p_1, q_1}^{s_1, \alpha}} \quad (5.1)$$

for $\lambda \leq 1$. By the local property of α -modulation and Triebel space, we actually have

$$\|f_\lambda\|_{L^{p_2}} \lesssim \|f_\lambda\|_{L^{p_1}},$$

and consequently $\lambda^{n(1-1/p_2)} \lesssim \lambda^{n(1-1/p_1)}$. Letting $\lambda \rightarrow 0^+$, we conclude

$$1/p_2 \leq 1/p_1. \quad (5.2)$$

Let g be a nonzero Schwartz function whose Fourier transform has compact support in $\{\xi : 3/4 \leq |\xi| \leq 4/3\}$. Set $\widehat{g_j}(\xi) := \widehat{g}(\xi/2^j)$. By the definition of Δ_j , we have $\Delta_j g_j = g_j$ for $j \geq 0$, and $\Delta_l g_j = 0$ if $l \neq j$. Denote

$$\Gamma_j = \{k \in \mathbb{Z}^n : \text{supp} \eta_k^\alpha \cap \{\xi : (3/4) \cdot 2^j \leq |\xi| \leq (4/3) \cdot 2^j\} \neq \emptyset\}. \quad (5.3)$$

We have $|\Gamma_j| \sim 2^{jn(1-\alpha)}$. For a truncated (only finite nonzero items) nonnegative sequence $\vec{a} = \{a_j\}_{j \in \mathbb{N}_0}$, we define

$$G^h := \sum_{j \in \mathbb{N}_0} a_j g_j^h, \quad g_j^h(x) := g_j(x + jhe_0), \quad (5.4)$$

where $h \in \mathbb{R}$, $e_0 = (1, 0, \dots, 0)$ is the unit vector in \mathbb{R}^n . By the definition of α -modulation space, we obtain that

$$\begin{aligned}
\|G^h\|_{M_{p_1, q_1}^{s_1, \alpha}} &= \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{s_1 q_1}{1-\alpha}} \|\square_k^\alpha G^h\|_{L^{p_1}}^{q_1} \right)^{\frac{1}{q_1}} \lesssim \left(\sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j} \langle k \rangle^{\frac{s_1 q_1}{1-\alpha}} \|a_j \square_k^\alpha g_j^h\|_{L^{p_1}}^{q_1} \right)^{\frac{1}{q_1}} \\
&\lesssim \left(\sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j} a_j^{q_1} \|\mathcal{F}^{-1} \eta_k^\alpha\|_{L^{p_1}}^{q_1} \langle k \rangle^{\frac{s_1 q_1}{1-\alpha}} \right)^{\frac{1}{q_1}} \sim \left(\sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j} a_j^{q_1} \langle k \rangle^{\frac{n\alpha q_1}{1-\alpha}(1-1/p_1)} 2^{js_1 q_1} \right)^{\frac{1}{q_1}} \\
&\sim \left(\sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j} a_j^{q_1} 2^{jq_1(n\alpha(1-1/p_1)+s_1)} \right)^{\frac{1}{q_1}} \sim \left(\sum_{j \in \mathbb{N}_0} |\Gamma_j| a_j^{q_1} 2^{jq_1(n\alpha(1-1/p_1)+s_1)} \right)^{\frac{1}{q_1}} \\
&= \left(\sum_{j \in \mathbb{N}_0} a_j^{q_1} 2^{jq_1(n\alpha(1-1/p_1)+s_1)} 2^{jn(1-\alpha)} \right)^{\frac{1}{q_1}} = \|\vec{a}\|_{l_{q_1}^{n(1-\alpha)/q_1+n\alpha(1-1/p_1)+s_1, 1}}.
\end{aligned} \tag{5.5}$$

On the other hand, by the same argument as in the proof of Proposition 4.1, we deduce that

$$\lim_{h \rightarrow \infty} \|G^h\|_{L^{p_2}} = \|\vec{a}\|_{l_{p_2}^{n(1-1/p_2), 1}}. \tag{5.6}$$

Since G^h is a Schwartz function, by the definition of local Hardy space, we have $\|G^h\|_{L^{p_2}} \lesssim \|G^h\|_{h^{p_2}}$. Thus,

$$\|\vec{a}\|_{l_{p_2}^{n(1-1/p_2), 1}} = \lim_{h \rightarrow \infty} \|G^h\|_{L^{p_2}} \lesssim \lim_{h \rightarrow \infty} \|G^h\|_{h^{p_2}}. \tag{5.7}$$

Combining (5.5) and (5.7), we have

$$\|\vec{a}\|_{l_{p_2}^{n(1-1/p_2), 1}} \lesssim \|\vec{a}\|_{l_{q_1}^{n(1-\alpha)/q_1+n\alpha(1-1/p_1)+s_1, 1}},$$

which implies the desired embedding $l_{q_1}^{n(1-\alpha)/q_1+n\alpha(1-1/p_1)+s_1, 1} \subset l_{p_2}^{n(1-1/p_2), 1}$.

Next, we turn to verify $l_{q_1}^{n\alpha(1-1/p_1)+s_1, \alpha} \subset l_{p_2}^{n\alpha(1-1/p_2), \alpha}$. Let $f \in \mathcal{S}$ be a nonzero smooth function whose Fourier transform has small support, such that $\square_k^\alpha f_k = f_k$ and $\square_l^\alpha f_k = 0$ if $k \neq l$, where we denote $\widehat{f}_k(x) = \widehat{f}\left(\frac{\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k}{\langle k \rangle^{\frac{1}{1-\alpha}}}\right)$. For a truncated (only finite nonzero items) nonnegative sequence $\vec{b} = \{b_k\}_{k \in \mathbb{Z}^n}$, we define

$$F^h(x) = \sum_{k \in \mathbb{Z}^n} b_k f_k^h, \quad f_k^h(x) = f_k(x - kh), \tag{5.8}$$

where $h \in \mathbb{R}$. Observing that F^h is a Schwartz function and $\|F^h\|_{L^{p_2}} \leq \|F^h\|_{h^{p_2}}$, we use the assumption to deduce

$$\|F^h\|_{L^{p_2}} \lesssim \|F^h\|_{M_{p_1, q_1}^{s_1, \alpha}}. \tag{5.9}$$

By the same argument as in the proof of Proposition 4.1, we obtain

$$\lim_{h \rightarrow \infty} \|F^h\|_{L^{p_2}} \sim \|\vec{b}\|_{l_{p_2}^{n\alpha(1-1/p_2), \alpha}}, \quad \|F^h\|_{M_{p_1, q_1}^{s_1, \alpha}} \sim \|\vec{b}\|_{l_{q_1}^{n\alpha(1-1/p_1)+s_1, \alpha}},$$

which implies

$$\|\vec{b}\|_{l_{p_2}^{n\alpha(1-1/p_2), \alpha}} \lesssim \|\vec{b}\|_{l_{q_1}^{n\alpha(1-1/p_1)+s_1, \alpha}}. \tag{5.10}$$

By the arbitrary of \vec{b} , we obtain the desired conclusion. \square

Now, we turn to verify the necessity of $M_{p_1, q_1}^{s_1, \alpha} \subset h^{p_2}$. Using Proposition 5.1, we obtain $1/p_2 \leq 1/p_1$ and the embedding relations:

$$l_{q_1}^{n(1-\alpha)/q_1+n\alpha(1-1/p_1)+s_1, 1} \subset l_{p_2}^{n(1-1/p_2), 1}, \quad l_{q_1}^{n\alpha(1-1/p_1)+s_1, \alpha} \subset l_{p_2}^{n\alpha(1-1/p_2), \alpha}.$$

By Lemma 2.4, $l_{q_1}^{n(1-\alpha)/q_1+n\alpha(1-1/p_1)+s_1, 1} \subset l_{p_2}^{n(1-1/p_2), 1}$ implies $s_1 \geq n\alpha(1/p_1 - 1/p_2) + n(1-\alpha)(1 - 1/p_2 - 1/q_1)$ and the inequality is strict if $1/p_2 > 1/q_1$. On the other hand, by Lemma 2.4 and

$l_{q_1}^{n\alpha(1-1/p_1)+s_1,\alpha} \subset l_{p_2}^{n\alpha(1-1/p_2),\alpha}$, we obtain $s_1 \geq n\alpha(1/p_1 - 1/p_2)$ for $1/p_2 \leq 1/q_1$, and $s_1 > n\alpha(1/p_1 - 1/p_2) + n(1-\alpha)(1/p_2 - 1/q_1)$ for $1/p_2 > 1/q_1$.

Combining with the above estimates, we obtain $s_1 \geq n\alpha(1/p_1 - 1/p_2) + (1-\alpha)B(p_2, q_1)$, with strict inequality for $1/p_2 > 1/q_1$.

5.2. Sufficiency of Theorem 1.2. As in the proof of Theorem 1.1, we actually only need to verify $M_{p_2, q_1}^{(1-\alpha)B(p_2, q_1), \alpha} \subset h^{p_2}$ for $1/p_2 \leq 1/q_1$, and $M_{p_2, q_1}^{(1-\alpha)B(p_2, q_1) + \epsilon, \alpha} \subset h^{p_2}$ for $1/p_2 > 1/q_1$, where ϵ is any positive number.

For $1/p_2 \leq 1/q_1$. We want to verify

$$M_{p_2, q_1}^{(1-\alpha)B(p_2, q_1), \alpha} \subset h^{p_2}. \quad (5.11)$$

Observing that $B(p_2, q_1) = 0$ and $M_{p_2, q_1}^{0, \alpha} \subset M_{p_2, \tilde{q}_1}^{0, \alpha}$ for $1/q_1 \geq (1 - 1/p_2) \vee 1/p_2$, where $1/\tilde{q}_1 = (1 - 1/p_2) \vee 1/p_2$, we only need to prove (5.11) for $1/p_2 \leq 1/q_1 \leq 1 - 1/p_2$ and $p_2 = q_1 \leq 2$.

By a dual argument, in $1/p_2 \leq 1/q_1 \leq 1 - 1/p_2$, (5.11) can be verified by the sufficiency of Theorem 1.1 proved before. Thus, we only need to verify (5.11) for $p_2 = q_1 \leq 2$. Observing that (5.11) has been verified for $p_2 = q_1 = 2$, by an interpolation argument, we only need to show (5.11) for $p_2 = q_1 \leq 1$.

In fact, when $p_2 = q_1 \leq 1$, we use the quasi-triangle inequality to deduce that

$$\|f\|_{h^{p_2}} = \left\| \sum_{k \in \mathbb{Z}^n} \square_k^\alpha f \right\|_{h^{p_2}} \leq \left(\sum_{k \in \mathbb{Z}^n} \|\square_k^\alpha f\|_{h^{p_2}}^{p_2} \right)^{1/p_2} \sim \left(\sum_{k \in \mathbb{Z}^n} \|\square_k^\alpha f\|_{L^{p_2}}^{p_2} \right)^{1/p_2} = \|f\|_{M_{p_2, q_1}^{0, \alpha}}, \quad (5.12)$$

which is just the embedding (5.11) for $p_2 = q_1 \leq 1$.

For $1/p_2 > 1/q_1$. We want to verify

$$M_{p_2, q_1}^{(1-\alpha)B(p_2, q_1) + \epsilon, \alpha} \subset h^{p_2}$$

for any fixed positive number ϵ . In fact, by the embedding $M_{p_2, p_2}^{(1-\alpha)B(p_2, p_2), \alpha} \subset h^{p_2}$ proved before, observing that

$$n(1-\alpha)/p_2 + (1-\alpha)B(p_2, p_2) < n(1-\alpha)/q_1 + (1-\alpha)B(p_2, q_1) + \epsilon,$$

we use Lemma 2.2 to deduce

$$M_{p_2, q_1}^{(1-\alpha)B(p_2, q_1) + \epsilon, \alpha} \subset M_{p_2, p_2}^{(1-\alpha)B(p_2, p_2), \alpha} \subset h^{p_2},$$

which is the desired conclusion.

6. EMBEDDING BETWEEN L^1 AND α -MODULATION SPACES

6.1. Proof of Theorem 1.3. We first give the proof for necessity. It is known that $h^p = L^p$ when $p > 1$, but $h^1 \subsetneq L^1$, due to the different structure between h^1 and L^1 , we have the a additional restriction associated with $L^1 \subset M_{p, q}^{s, \alpha}$.

Proposition 6.1. *Let $0 < p \leq \infty$, $0 < q < \infty$, $s \in \mathbb{R}$, then $L^1 \subset M_{p, q}^{s, \alpha}$ implies*

$$s < n\alpha(1/p - 1) + n(1-\alpha)A(1, q)$$

Proof. Take f to be a smooth function satisfying that $\text{supp } \hat{f} \subset B(0, 2)$ and $\hat{f}(\xi) = 1$ on $B(0, 1)$. Denote $\hat{f}_j(\xi) = \hat{f}(\xi/2^j)$ for $j \geq 1$. By the scaling of L^1 , we obtain $\|f_j\|_{L^1} \sim 1$ for all j . However, by the assumption of f_j , for any finite subset of \mathbb{Z}^n , denoted by A , we can find a sufficiently large J such that

$$\begin{aligned} \left(\sum_{k \in A} \langle k \rangle^{\frac{sq}{1-\alpha}} \langle k \rangle^{\frac{n\alpha}{1-\alpha}(1-1/p)q} \right)^{\frac{1}{q}} &\sim \left(\sum_{k \in A} \langle k \rangle^{\frac{sq}{1-\alpha}} \|\mathcal{F}^{-1} \eta_k^\alpha\|_{L^p}^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{sq}{1-\alpha}} \|\square_k^\alpha f_J\|_{L^p}^q \right)^{\frac{1}{q}} \sim \|f_J\|_{M_{p, q}^{s, \alpha}} \lesssim \|f_J\|_{L^1} \lesssim 1. \end{aligned}$$

By the arbitrary of A , we actually obtain

$$\left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{sq}{1-\alpha}} \langle k \rangle^{\frac{n\alpha}{1-\alpha}(1-1/p)q} \right)^{\frac{1}{q}} \lesssim 1.$$

which yields that $s < n\alpha(1/p - 1) + n(1 - \alpha)(-1/q) = n\alpha(1/p - 1) + n(1 - \alpha)A(1, q)$. \square

By the same method used to prove Proposition 4.1, we obtain

$$s \leq n\alpha(1/p - 1) + n(1 - \alpha)A(1, q). \quad (6.1)$$

Then, we use Proposition 6.1 to conclude that inequality (6.1) must be strict when $q \neq \infty$.

Next, we turn to the sufficiency part. When $q = \infty$, we have $p \geq 1$ and $s \leq n\alpha(1/p - 1)$. Using Young's inequality, we deduce that

$$\begin{aligned} \|f\|_{M_{p,\infty}^{s,\alpha}} &= \sup_{k \in \mathbb{Z}^n} \|\square_k^\alpha f\|_{L^p} \langle k \rangle^{\frac{s}{1-\alpha}} \lesssim \sup_{k \in \mathbb{Z}^n} \|\mathcal{F}^{-1} \eta_k^\alpha\|_{L^p} \langle k \rangle^{\frac{s}{1-\alpha}} \|f\|_{L^1} \\ &\lesssim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{n\alpha}{1-\alpha}(1-1/p)} \langle k \rangle^{\frac{s}{1-\alpha}} \|f\|_{L^1} \lesssim \|f\|_{L^1}. \end{aligned} \quad (6.2)$$

For $q < \infty$, taking ϵ to be a fixed positive number, observing that $n(1 - \alpha)/q + n\alpha(1/p - 1) + n(1 - \alpha)A(1, q) - \epsilon < n\alpha(1/p - 1)$, we use Lemma 2.2 to deduce

$$\|f\|_{M_{p,q}^{n\alpha(1/p-1)+n(1-\alpha)A(1,q)-\epsilon,\alpha}} \lesssim \|f\|_{M_{p,\infty}^{n\alpha(1/p-1),\alpha}} \lesssim \|f\|_{L^1}. \quad (6.3)$$

6.2. Proof of Theorem 1.4. By the same method used in the proof of Proposition 5.1, we can verify the necessity of Theorem 1.4. On the other hand, using Theorem 1.2 and the fact that $h^1 \subset L^1$, we get the sufficiency of Theorem 1.4.

7. EMBEDDING BETWEEN L^∞ AND α -MODULATION SPACES

7.1. Proof of Theorem 1.5. First, we give the proof for necessity. By the same method used in the proof Proposition 4.1, we deduce $1/p \leq 1/\infty$, which yields $p = \infty$. We also have $s \leq n(1 - \alpha)A(\infty, q)$ with strict inequality when $q \neq \infty$.

Next, we turn to the sufficiency part. For $q = \infty$, we have $s \leq 0$. Thus, we have

$$\|f\|_{M_{\infty,\infty}^{s,\alpha}} \leq \|f\|_{M_{\infty,\infty}^{0,\alpha}} = \sup_{k \in \mathbb{Z}^n} \|\square_k^\alpha f\|_\infty \leq \sup_{k \in \mathbb{Z}^n} \|f\|_\infty \leq \|f\|_{L^\infty}.$$

When $q \neq \infty$ and $s < -n(1 - \alpha)/q$, by Young's inequality, we have

$$\|f\|_{M_{\infty,q}^{s,\alpha}} = \left(\sum_{k \in \mathbb{Z}^n} \|\square_k^\alpha f\|_{L^\infty}^q \langle k \rangle^{\frac{sq}{1-\alpha}} \right)^{\frac{1}{q}} \lesssim \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{sq}{1-\alpha}} \right)^{\frac{1}{q}} \|f\|_{L^\infty} \lesssim \|f\|_{L^\infty}, \quad (7.1)$$

where we use $sq/(1 - \alpha) < -n$ in the last inequality.

7.2. Proof of Theorem 1.6. First, we give the proof for necessity. By the structure of L^∞ , we establish the following proposition for restriction of $M_{p,q}^{s,\alpha} \subset L^\infty$.

Proposition 7.1. *Let $0 < p \leq \infty$, $q > 1$, then $M_{p,q}^{s,\alpha} \subset L^\infty$ implies*

$$l_q^{s+n\alpha(1-1/p),\alpha} \subset l_1^{n\alpha,\alpha}.$$

Proof. Take $f \in \mathcal{S}$ to be a nonzero smooth function whose Fourier transform has small support, such that $f(0) = 1$, $\square_k^\alpha f_k = f_k$ and $\square_l^\alpha f_k = 0$ if $k \neq l$, where we denote $\widehat{f}_k(x) = \widehat{f}\left(\frac{\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k}{\langle k \rangle^{\frac{\alpha}{1-\alpha}}}\right)$. For a truncated (only finite nonzero items) nonnegative sequence $\vec{a} = \{a_k\}_{k \in \mathbb{Z}^n}$, we define

$$F(x) = \sum_{k \in \mathbb{Z}^n} a_k f_k. \quad (7.2)$$

By a direct computation, we have

$$\|F\|_{M_{p_2,q_2}^{s_2,\alpha}} \sim \|\vec{a}\|_{l_{q_2}^{n\alpha(1-1/p_2)+s_2,\alpha}}.$$

On the other hand, observing that $F(x) = \sum_{k \in \mathbb{Z}^n} a_k \langle k \rangle^{\frac{\alpha n}{1-\alpha}} e^{2\pi i \langle k \rangle^{\frac{\alpha}{1-\alpha}} k \cdot x} f(\langle k \rangle^{\frac{\alpha}{1-\alpha}} x)$, we have

$$\|F\|_{L^\infty} \geq F(0) = \sum_{k \in \mathbb{Z}^n} a_k \langle k \rangle^{\frac{\alpha n}{1-\alpha}} f(0) = \sum_{k \in \mathbb{Z}^n} a_k \langle k \rangle^{\frac{\alpha n}{1-\alpha}} = \|\vec{a}\|_{l_1^{n\alpha, \alpha}}.$$

Thus, we have

$$\|\vec{a}\|_{l_1^{n\alpha, \alpha}} \lesssim \|F\|_{L^\infty} \lesssim \|F\|_{M_{p,q}^{s,\alpha}} \sim \|\vec{a}\|_{l_q^{n\alpha(1-1/p)+s,\alpha}}.$$

By the arbitrary of \vec{a} , we obtain the desired conclusion. \square

Using Proposition 7.1 and Lemma 2.4, we deduce that $s \geq n\alpha/p$ for $1 \leq 1/q$, and $s > n\alpha/p + n(1-\alpha)(1-1/q)$ for $1 > 1/q$, which is just the conclusion.

Now, we turn to the proof of sufficiency. In fact, by Lemma 2.2, we have that

$$\|f\|_{L^\infty} \leq \sum_{k \in \mathbb{R}^n} \|\square_k^\alpha f\|_{L^\infty} = \|f\|_{M_{\infty,1}^{0,\alpha}} \lesssim \|f\|_{M_{p,q}^{s,\alpha}}. \quad (7.3)$$

where $s \geq n\alpha/p$ for $1 \leq 1/q$, or $s > n\alpha/p + n(1-\alpha)(1-1/q)$ for $1 > 1/q$.

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